Bessel's Inequality for finite orthonormal Sets:

Let $\{e_1, e_2, .., e_n\}$ be a finite orthonormal set in a Hilbert Space H. If x is any vector in H, then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2 \quad \text{---(1)}$$

Further $\quad x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j \quad \text{---(2)}$

for each 'j'.

Pf:- Consider the vector —

$$y = x - \sum_{i=1}^{n} (x, e_i) e_i$$

We have —

$$\|y\|^2 = (y, y) = \left(x - \sum_{i=1}^{n} (x, e_i) e_i ,\ x - \sum_{j=1}^{n} (x, e_j) e_j \right)$$

$$= (x, x) - \sum_{i=1}^{n} (x, e_i)(e_i, x)$$

$$- \sum_{j=1}^{n} \overline{(x, e_j)} (x, e_j)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i) \overline{(x, e_j)} (e_i, e_j)$$

$$= \|x\|^2 - \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)} - \sum_{j=1}^{n} \overline{(x, e_j)} (x, e_j)$$

$$+ \sum_{i=1}^{n} (x, e_i) \overline{(x, e_i)}$$

, summing w.r.t. 'j' with $(e_i, e_j) = 1$ when $j = i$ & $(e_i, e_j) = 0$ when $j \neq i$

$$= \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 + \sum_{i=1}^{n} |(x, e_i)|^2$$

$$\|y\|^2 = \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 \quad \to (3)$$

Now $\|y\|^2 \geq 0$

$\therefore$ From (3) →

$$\|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2 \geq 0$$

$$\therefore \sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$$

which proves result (1).

Now to prove result (2).

For each j ; $1 \leq j \leq n$, we have—

$$\left(x - \sum_{i=1}^{n} (x, e_i) e_i , e_j\right)$$

$$= (x, e_j) - \left(\sum_{i=1}^{n} (x, e_i) e_i , e_j\right)$$

$$= (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i, e_j)$$

$$= (x, e_j) - (x, e_j)$$

, since $(e_i, e_j) = 1$ when $i = j$ and $(e_i, e_j) = 0$ for $i \neq j$

$$= 0$$

$\therefore$ $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j!

(Proved)

Complete orthonormal set

An orthonormal set is said to be complete if it is not contained in any larger orthonormal set.

Thm. (A characterization theorem for complete orthonormal sets.)

Let H be a H.S. and let $\{e_i\}$ be an orthonormal set in H. Then the following conditions are equivalent to one another:-

(i) $\{e_i\}$ is complete

(ii) $x \perp \{e_i\} \Rightarrow x = 0$

(iii) if x is an arbitrary vector in H, then $x = \sum (x, e_i) e_i$

(iv) if x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$

Pf → (i) ⇒ (ii).

Given that the orthonormal set $\{e_i\}$ is complete and we have to prove that $x \perp \{e_i\} \Rightarrow x = 0$.

Suppose $x \perp \{e_i\}$ and $x \neq 0$.

Then $e = \dfrac{x}{\|x\|}$ is a unit vector s.t.

$e \perp \{e_i\}$ i.e. $(e, e_i) = 0$ for each i.

$\therefore$ $\{e, e_i\}$ is an orthonormal set

which properly contains $\{e_i\}$.
But this contradicts the hypothesis that $\{e_i\}$ is a complete orthonormal set. $\therefore$ $x \perp \{e_i\} \Rightarrow x = 0$

(ii) $\Rightarrow$ (iii) Given that $\rightarrow$

$$x \perp \{e_i\} \Rightarrow x = 0$$

To show if x is an arbitrary vector in H, then $x = \sum (x, e_i) e_i$.
We know if $\{e_i\}$ is an orthonormal set in a H.S. H and if x is an arb. vector in H, then –

$$x - \sum (x, e_i) e_i \perp e_j \quad \text{for each } j.$$

i.e. $x - \sum (x, e_i) e_i \perp \{e_i\}$

$\therefore$ by hypothesis $x \perp \{e_i\} \Rightarrow x = 0$, we have $\rightarrow$

$$x - \sum (x, e_i) e_i = 0$$

$$\Rightarrow \quad x = \sum (x, e_i) e_i$$

(iii) $\Rightarrow$ (iv)

Given that for any vector 'x' in H, we have $\rightarrow$ $x = \sum (x, e_i) e_i$.
To prove that $\|x\|^2 = \sum |(x, e_i)|^2$
we have $\rightarrow$ $\|x\|^2 = (x, x)$

$$= \left(\sum_i (x, e_i) e_i , \sum_j (x, e_j) e_j \right)$$

$$= \sum_i \sum_j (x, e_i) \overline{(x, e_j)} (e_i, e_j)$$

$$= \sum_i (x, e_i) \overline{(x, e_i)}$$

, on summing w.r.t. j!

$$\|x\|^2 = \sum_i |(x, e_i)|^2$$

<u>(iv) ⇒ (i)</u> Given if x is any arbitrary vector in H, Then —

$$\|x\|^2 = \sum |(x, e_i)|^2$$

To prove that $\{e_i\}$ is complete.

Suppose $\{e_i\}$ is not complete.

~~By hypothesis, we have~~

Then $\{e_i\}$ is a proper subset of an orthonormal set $\{e_i, e\}$.

By hypothesis, we have —

$$\|e\|^2 = \sum |(e, e_i)|^2$$

$$= 0, \quad \because e \perp e_i \text{ for each } i!$$

Now $\|e\|^2 = 0$ contradicts the fact that e is a unit vector.

$\therefore$ The orthonormal set $\{e_i\}$ must be complete.

(Proved)